Due Sun

1.8 – Introduction to Linear Transformations

$\boldsymbol{R}^2$ is the set of all ordered pairs, $\boldsymbol{R}^3$ is the set of all ordered triples, and $\boldsymbol{R}^n$ is the set of all ordered n-tuples. Elements of R^n are called **vectors**.

The **standard basis vectors** for R^n are

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

These can also be written as row vectors if convenient.

All vectors in $\mathbb{R}^n$ can be written as linear combinations of these vectors.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \cdots + x_n \vec{e}_n$$

A **function** f is a rule that associates with an input (an element of the **domain** of f) a unique output (an element of the **codomain** of f). The output is the **image** of the input, and the set of all images is called the **range** of f. Note that the range of f is a subset of the codomain of f.

$f(a) = b \leftarrow$ image of a under f

$a \in$ domain of f, $b \in$ codomain of f

Consider $f(x) = |x|$

Dom $f: \mathbb{R}$

Codom. $f: \mathbb{R}$

Range $f: [0, \infty)$

The range is a subset of the codomain.

Transformations

Consider the system
$$4 = 2x - 3y + 5z$$
$$6 = 7x + 2y - z$$

matrix form: $\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 5 \\ 7 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

The 2×3 matrix $A = \begin{bmatrix} 2 & -3 & 5 \\ 7 & 2 & -1 \end{bmatrix}$ maps vectors in $\mathbb{R}^3$ into vectors in $\mathbb{R}^2$.

A is called a matrix transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.

Definition 1: If T is a function with domain R^n and codomain R^m, then we say that T is a **transformation** from R^n to R^m or that T **maps** from R^n to R^m, which we denote by writing $T: R^n \to R^m$. In the special case where $m = n$, a transformation is sometimes called an **operator** on R^n.

$A \vec{x} = \vec{w} = T_A(\vec{x})$ in R^3 R^2

$$\begin{bmatrix} 3 & 1 & 2 \\ 6 & 7 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \end{bmatrix} \qquad T_A\left(\begin{bmatrix} 4 \\ -5 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$$

$T_A : R^3 \to R^2$

5. Find the domain and codomain of the transformation defined by the matrix product.

a. $\underset{2\times 3}{\begin{bmatrix} 3 & 1 & 2 \\ 6 & 7 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}$

Dom: R^3

Codomain: R^2

b. $\underset{3\times 2}{\begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 2 & -5 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} 2x_1 - x_2 \\ 4x_1 + 3x_2 \\ 2x_1 - 5x_2 \end{bmatrix}$

Dom: R^2

Codomain: R^3

A **matrix transformation** $\mathbf{w} = A\mathbf{x}$ maps a vector $\mathbf{x} \in R^n$ to a vector $\mathbf{w} \in R^m$ by multiplying $\mathbf{x}$ on the left by A [which is an $m \times n$ matrix]. If $m = n$, then we call the transformation a **matrix operator**. A matrix transformation is denoted $T_A: R^n \to R^m$ or $\mathbf{w} = T_A(\mathbf{x})$ if we do not need to specify the domain and codomain. This can also be written in the form

$$\mathbf{x} \xrightarrow{T_A} \mathbf{w}$$

verbalized as "T_A maps $\mathbf{x}$ into $\mathbf{w}$." The matrix A is the **standard matrix** for the transformation.

8. Find the domain and codomain of the transformation T defined by the formula.

a. $T(x_1, x_2, x_3, x_4) = (x_1, x_2)$

b. $T(x_1, x_2, x_3) = (x_1, x_2 - x_3, x_2)$

a) Dom: R^4

Codom: R^2

b) Dom: R^3

Codom: R^3] operator

12. Find the standard matrix for the transformation defined by the equations.

a.

$w_1 = -x_1 + x_2$
$w_2 = 3x_1 - 2x_2$
$w_3 = 5x_1 - 7x_2$

b.

$w_1 = x_1$
$w_2 = x_1 + x_2$
$w_3 = x_1 + x_2 + x_3$
$w_4 = x_1 + x_2 + x_3 + x_4$

$$A = \begin{bmatrix} -1 & 1 \\ 3 & -2 \\ 5 & -7 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Note: $A\vec{e}_1 = \begin{bmatrix} -1 & 1 \\ 3 & -2 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$ $\quad A\vec{e}_2 = \begin{bmatrix} 1 \\ -2 \\ -7 \end{bmatrix}$

13. Find the standard matrix for the transformation T defined by the formula.

a. $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$
b. $T(x_1, x_2, x_3) = (4x_1 + x_2, 5x_1 + x_2)$

a. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} \Rightarrow [T] = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

b. $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 + x_2 \\ 5x_1 + x_2 \end{bmatrix} \Rightarrow [T] = \begin{bmatrix} 4 & 1 & 0 \\ 5 & 1 & 0 \end{bmatrix}$

The **zero transformation** from R^n to R^m, $T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$, maps every vector in R^n to the zero vector in R^m.

The **identity operator** $T_{I_n}(\mathbf{x}) = I_n(\mathbf{x}) = \mathbf{x}$ maps every vector in R^n to itself.

Theorem 1.8.1 Properties of Matrix Transformations

For every matrix A the matrix transformation $T_A: R^n \to R^m$ has the following properties for all vectors $\mathbf{u}$ and $\mathbf{v}$ and for every scalar k:

a) $T_A(\mathbf{0}) = \mathbf{0}$

b) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ (homogeneity property)

c) $T_A(\mathbf{u}+\mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ (additivity property)

d) $T_A(\mathbf{u}-\mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$

True because of properties of matrix arithmetic

Theorem 1.8.2 $T: R^n \to R^m$ is a matrix transformation if and only if the following relationships hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ and for every scalar k:

i) $T_A(\mathbf{u}+\mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ (additivity property)

ii) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ (homogeneity property)

$\Rightarrow$ True from 1.8.1

pf: ($\Leftarrow$) Assume (i) & (ii) hold. The proof will be complete if there is an $m \times n$ matrix A such that $T_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in R^n$.

By (i) & (ii), $T(k_1\vec{u}_1 + k_2\vec{u}_2 + \cdots + k_n\vec{u}_n)$

$$= k_1 T(\vec{u}_1) + k_2 T(\vec{u}_2) + \cdots + k_n T(\vec{u}_n)$$

for all scalars k_i and $\vec{u}_i \in R^n$.

In particular, this holds for $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$.

We consider scalars $x_1, x_2, \dots, x_n$.

This becomes $x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2) + \cdots + x_n T(\vec{e}_n)$

$$= \left[T(\vec{e}_1) \mid T(\vec{e}_2) \mid \cdots \mid T(\vec{e}_n) \right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= A\vec{x}.$$

We've shown that for $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$,

$$T(\vec{x}) = A\vec{x}.$$

22. Use Theorem 1.8.2 to show that T is a matrix transformation.

a. $T(x, y, z) = (x + y, y + z, x)$

b. $T(x_1, x_2, x_3) = (x_1, x_3, x_1 + x_2)$

i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

ii) $T(k\vec{u}) = kT(\vec{u})$

a. Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ and $k \in \mathbb{R}$.

i) $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

$$T(\vec{u}+\vec{v}) = (u_1+v_1+u_2+v_2,\ u_2+v_2+u_3+v_3,\ u_1+v_1)$$

$$= (u_1+u_2+v_1+v_2,\ u_2+u_3+v_2+v_3,\ u_1+v_1)$$

$$= (u_1+u_2,\ u_2+u_3,\ u_1) + (v_1+v_2,\ v_2+v_3,\ v_1)$$

$$= T(\vec{u}) + T(\vec{v}) \quad ✓$$

ii) $k\vec{u} = (ku_1, ku_2, ku_3)$

so $T(k\vec{u}) = (ku_1 + ku_2,\ ku_2 + ku_3,\ ku_1)$

$= k(u_1 + u_2,\ u_2 + u_3,\ u_1)$

$= k\,T(\vec{u})$ ✓

These can be combined: $T(k\vec{u} + \vec{v}) = kT(\vec{u}) + T(\vec{v})$

23. Use Theorem 1.8.2 to show that T is not a matrix transformation.

a. $T(x, y) = (x^2, y)$

b. $T(x, y, z) = (x, y, xz)$

a. Let $\vec{x} = (x, y)$, $k \in R$.

$T(kx, ky) = (k^2x^2, ky)$

$= k(kx^2, y) \neq kT(x, y)$ ✗

Vocab

A **linear transformation** $T: R^n \to R^m$ possesses the two **linearity conditions**, those being:

For all vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n and for every scalar k,

i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity property)

ii) $T(k\mathbf{u}) = kT(\mathbf{u})$ (homogeneity property)

Theorem 1.8.3 Every linear transformation from R^n to R^m is a matrix transformation, and conversely every matrix transformation from R^n to R^m is a linear transformation.

we can use these interchangably

28. The images of the standard basis vectors for R^3 are given for a linear transformation $T: R^3 \to R^3$. Find the standard matrix for the transformation, and find $T(\mathbf{x})$.

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, T(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}; \mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{[T_A] = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix} = A}$$

$$T_A(\vec{x}) = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 8 \end{bmatrix}$$

Suppose we know that for vectors $\vec{u}$ & $\vec{v}$, $\underline{\vec{e}_1} = \underline{a\vec{u} + b\vec{v}}$. If T is a linear transf., then $T(\underline{\vec{e}_1}) = T(\underline{a\vec{u}+b\vec{v}}) = aT(\vec{u}) + bT(\vec{v})$

Ex: Find the standard matrix A for the linear transformation

$T: R^2 \to R^3$ for which $T\left(\underbrace{\begin{bmatrix}-1\\2\end{bmatrix}}_{\vec{u}}\right) = \begin{bmatrix}2\\1\\0\end{bmatrix}$ and $T\left(\underbrace{\begin{bmatrix}3\\-5\end{bmatrix}}_{\vec{v}}\right) = \begin{bmatrix}5\\-7\\1\end{bmatrix}$ and use it

to compute $T\left(\begin{bmatrix}-4\\3\end{bmatrix}\right)$.

we want a & b such that $a\vec{u}+b\vec{v}=\vec{e}_1$

and another a & b such that $a\vec{u}+b\vec{v}=\vec{e}_2$

$$a\begin{bmatrix}-1\\2\end{bmatrix}+b\begin{bmatrix}3\\-5\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix} \quad \left(=\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$\begin{bmatrix}-1 & 3\\2 & -5\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix} \quad \left(=\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$\left[\begin{array}{cc|cc}-1 & 3 & 1 & 0\\2 & -5 & 0 & 1\end{array}\right] \to \left[\begin{array}{cc|cc}1 & 0 & 5 & 3\\0 & 1 & 2 & 1\end{array}\right]$$

$\vec{e}_1 = 5\vec{u}+2\vec{v}$ $\qquad$ $\vec{e}_2 = 3\vec{u}+1\vec{v}$

$T(\vec{e}_1) = 5T(\vec{u})+2T(\vec{v})$ $\qquad$ $T(\vec{e}_2) = 3T(\vec{u})+T(\vec{v})$

$T(\vec{e}_1) = 5\begin{bmatrix}2\\1\\0\end{bmatrix}+2\begin{bmatrix}5\\-7\\1\end{bmatrix}$ $\qquad$ $T(\vec{e}_2) = 3\begin{bmatrix}2\\1\\0\end{bmatrix}+\begin{bmatrix}5\\-7\\1\end{bmatrix}$

$T(\vec{e}_1) = \begin{bmatrix}20\\-9\\2\end{bmatrix}$ $\qquad$ $T(\vec{e}_2) = \begin{bmatrix}11\\-4\\1\end{bmatrix}$

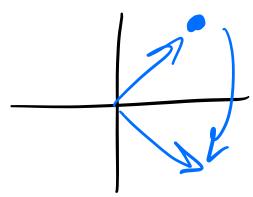
$T_A: R^2 \to R^3$

$$A = \begin{bmatrix} 20 & 11 \\ -9 & -4 \\ 2 & 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} -4 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 20 & 11 \\ -9 & -4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -47 \\ 24 \\ -5 \end{bmatrix}$$

Theorem 1.8.4 If $T_A: R^n \to R^m$ and $T_B: R^n \to R^m$ are matrix transformations, and if
$T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector $\mathbf{x}$ in R^n, then $A = B$.

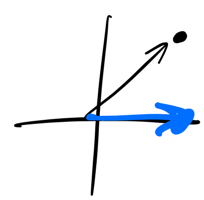
Matrix operators on R^2



Reflection operators

- Reflection about the x-axis: $T(x, y) = T(x, -y)$, $T_A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- Reflection about the y-axis: $T(x, y) = T(-x, y)$, $T_A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
- Reflection about the line $y = x$: $T(x, y) = (y, x)$, $T_A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Projection operators



- Orthogonal projection onto the x-axis: $T(x, y) = T(x, 0)$,
  $T_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
- Orthogonal projection onto the y-axis: $T(x, y) = T(0, y)$,
  $T_A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Rotation operator

- Counterclockwise rotation about the origin through an angle θ:

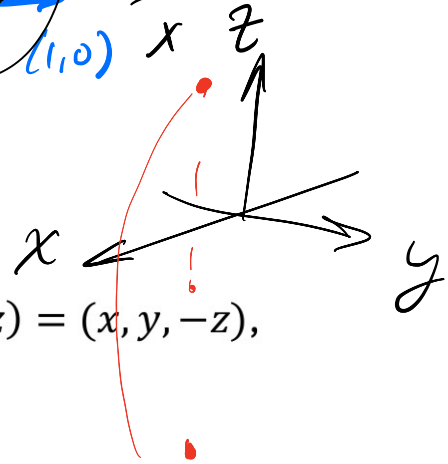
$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

θ $90°$

$\cos(\alpha+\beta) =$

$\cos\alpha\cos\beta - \sin\alpha\sin\beta$

$T(\vec{e}_1)$ $T(e_2)$



Matrix operators on R^3

Reflection operators

- Reflection about the xy-plane: $T(x, y, z) = (x, y, -z)$,

$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- Reflection about the xz-plane: $T(x, y, z) = (x, -y, z)$,

$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Reflection about the yz-plane: $T(x, y, z) = (-x, y, z)$,

$$T_A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Projection operators

- Orthogonal projection onto the xy-plane: $T(x, y, z) = (x, y, 0)$,

$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Orthogonal projection onto the xz-plane: $T(x, y, z) = (x, 0, z)$,

$$T_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Orthogonal projection onto the yz-plane: $T(x, y, z) = (0, y, z)$,

$$T_A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$